Complex Analysis (for Physics)

Midterm Exam

December 9, 2022 (18:30 – 20:30)

university of groningen

Please turn over and read the instructions!

1) Prove that if $|z| = 1$ then $\left|\dfrac{\overline{b}z + \overline{a}}{az + b}\right| = 1$ for all complex numbers a, b, $(a, b) \neq (0, 0)$.

2) Find all complex number solutions of the equation $z^2 + |z| = 0$. Write your final answer in algebraic form.

3) Show that the complex function $w = z + \dfrac{1}{z}$ maps the circles $|z| = r$ (with $r > 1$) onto ellipses. What happens when $r \to 1$?

4) Consider the complex function $f(x + iy) = (x^2 + 2y) + i(x^2 + y^2)$ and determine the points $z_0 \in \mathbb{C}$ at which the derivative $f'(z_0)$ exists.

5) Determine the points at which the complex function $g(z) = \dfrac{1}{(1 - \sin z)^2}$ has no derivative and compute its derivative where it exists.

6) Verify that the function $v(x, y) = y + e^{x^2 - y^2} \sin 2xy$ is harmonic in $\mathbb{C}$ and find a harmonic conjugate $-u(x, y)$ such that $u(0, 0) = 3$.

Formula sheet

Numbers: $z = x + iy$ (algebraic form), $x, y \in \mathbb{R}$, $i^2 = -1$, $\overline{z} = x - iy$

Real and imaginary parts: $x = \mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}$, $y = \mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}$

Basic operations: If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$, $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)$

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}, \quad z_2 \neq 0$$

Polar form: $z = r(\cos\theta + i \sin\theta)$, $r \geq 0$, $\theta \in (-\pi, \pi]$

Modulus: $r = |z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$

Argument: $\theta = \mathrm{Arg}(z)$ (principal value), $\arg(z) = \mathrm{Arg}(z) + 2\pi k$, $k \in \mathbb{Z}$

Identities: $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$, $|\overline{z}| = |z|$, $\arg(\overline{z}) = -\arg(z)$

$|z_1 z_2| = |z_1||z_2|$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Triangle inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$

De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$, $n \in \mathbb{Z}$

Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

Exponential form: $z = re^{i\theta}$

Functions: $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$

Complex exponential: $e^z := 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$e^{z_1 + z_2} = e^{z_1} e^{z_2}$, $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$

Trigonometric functions: $\cos z := \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin z := \dfrac{e^{iz} - e^{-iz}}{2i}$

Complex Logarithm: $\mathrm{Log}\, z := \mathrm{Log}\,|z| + i\,\mathrm{Arg}\, z$ (principal value)

Derivatives: $f'(z_0) = \displaystyle\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$

Cauchy-Riemann equations: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Laplace's equation: $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$, solutions are called *harmonic*

Instructions

- **write your name and student number on the envelope and on the top of each sheet of writing paper!**
- use the writing (lined) and scratch (blank) paper provided, raise your hand if you need more paper
- start each question on a new page
- use a pen with black or blue ink
- do not use any kind of correcting fluid or tape
- any rough work should be crossed through neatly so it can be seen
- this is a closed-book exam, you are not allowed to use the textbook or your notes or any other written material
- you are allowed to use the formula sheet provided or a simple pocket calculator
- programmable/graphing calculators are not allowed, nor the use of electronic devices (tablet, laptop, phone, etc.) to solve the exercises
- your work should be clearly and logically structured
- **explain your reasoning using words**
- show all your calculations, an answer without any computation will not be rewarded
- each problem is worth 15 points
- you get 10 free points
- upon completion[1] place your worksheets in the envelope and submit them at the front desk

[1] At the end of the exam or after you finished whichever is sooner.